JOURNAL OF COMPUTATIONAL PHYSICS 95, 212–227 (1991)

The Discrete Continuity Equation in Primitive Variable Solutions of Incompressible Flow

F. SOTIROPOULOS AND S. ABDALLAH

Department of Aerospace Engineering and Engineering Mechanics, University of Cincinnati, Cincinnati, Ohio 45221

Received December 12, 1989; revised April 20, 1990

The use of a non-staggered computational grid for the numerical solutions of the incompressible flow equations has many advantages over the use of a staggered grid. A penalty, however, is inherent in the finite-difference approximations of the governing equations on non-staggered grids. In the primitive-variable solutions, the penalty is that the discrete continuity equation does not converge to machine accuracy. Rather it converges to a source term which is proportional to the fourth-order derivative of the pressure, the time increment, and the square of the grid spacing. An approach which minimizes the error in the discrete continuity equation is developed. Numerical results obtained for the driven cavity problem confirm the analytical developments.

INTRODUCTION

There are two common formulations for the numerical solution of the incompressible Navier–Stokes equations in primitive variables; the artificial compressibility and the pressure Poisson equation methods. In both methods, the velocity field is calculated from the time dependent momentum equation using time marching techniques, while each method employs a different equation to compute the pressure.

In the artificial compressibility method, a time derivative of the pressure is added to the continuity equation [1] and the incompressible field is treated as compressible during the transient calculations. On the other hand, the pressure Poisson method [2] replaces the continuity equation with a second-order elliptic Poisson equation for the pressure.

An important issue in the numerical solutions of the primitive variable formulations is the satisfaction of the discrete continuity equation when staggered or non-staggered computational grids are used. It has been shown for both methods, the artificial compressibility and the pressure Poisson, that the discrete continuity equation is satisfied to machine zero on staggered grids [2, 3]. Unfortunately, this is not true on non-staggered grids. The artificial compressibility method requires the explicit addition of a fourth-order artificial dissipation term to the discrete continuity equation to eliminate odd–even decoupling in the pressure field. The

0021-9991/91 $3.00

odd–even decoupling is caused by central second-order finite-difference approximations of the first-order continuity equation. Therefore the discrete divergence of the velocity field is not driven to machine zero but rather to a term proportional to the fourth-order derivative of the pressure [4]. Similarly, the pressure Poisson formulation may not satisfy the discrete continuity equation exactly on non-staggered grids. This phenomenon can be explained by investigating the discrete pressure equation which can be obtained by:

1. Discretization of the continuum pressure Poisson equation using central second-order accurate formulas [5, 6].

2. Direct derivation from the discrete divergence of the discrete momentum equation [2].

Although the above approaches lead to the same discrete pressure equation on a staggered grid, they do not give the same equation on non-staggered grids.

Careful examination of the two forms of the discrete pressure equation shows that the direct derivation of the discrete pressure equation satisfies the discrete continuity exactly but fails to give a smooth pressure field. The oscillatory behavior of the pressure is caused by the odd–even decoupling inherent in the resulting discrete pressure equation. On the other hand, the first approach gives a smooth pressure field, but has two major problems:

1. The compatibility condition of the Poisson Neumann problem is not automatically satisfied.

2. The discrete continuity equation is not exactly satisfied.

The first problem has been resolved in Refs. [5–9]. The use of the consistent finite-difference method of Abdallah [5, 6] satisfies the compatibility condition exactly on non-staggered grids. While in Refs. [7–9] a uniform correction for the source term of the pressure Poisson equation is employed in order to satisfy the compatability condition. The second problem is the subject of this paper. More specifically, we discuss the reasons which prevent the discrete divergence of the velocity from going to zero and we estimate the error in the discrete continuity equation. Finally, we propose an optimum discrete form for the pressure equation which minimizes the error in the continuity equation and provides a smooth pressure field.

MATHEMATICAL FORMULATION

Governing Equations

The equations which govern the laminar, incompressible flow of a Newtonian fluid are given in Cartesian coordinates as follows:

Continuity,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

x-Momentum,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

y-Momentum,

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{3}$$

where u and v are the velocity components in the x- and y-directions, respectively, P is the static pressure divided by the density, and Re is the Reynolds number.

The main difficulty associated with the solution of the system (1)–(3) is the continuity equation (1). Equation (1) is a constraint which the velocity field has to satisfy at any instant in time and not an evolution equation of the type (2) or (3). Also, Eq. (1) does not involve the pressure, which appears only in the momentum equations (2) and (3).

A numerical solution for Eqs. (1)–(3) must incorporate a procedure for taking into account the important interaction between the pressure and velocity fields. In order to achieve this coupling between the pressure and the velocity fields, the continuity equation (1) must be replaced by an equation which involves both pressure and velocity and at the same time guarantees the satisfaction of the incompressibility constraint.

The Pressure Poisson Formulation

In the pressure Poisson formulation a second-order elliptic equation, of Poisson type for the pressure, is derived by applying the divergence operator to the momentum equation:

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}=-\left(\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}\right)+\frac{\partial D}{\partial t}, \tag{4}$$

where

$$\xi=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{4a}$$

$$\eta=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{4b}$$

$$D=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}. \tag{4c}$$

Up to this point the continuity equation (1) has not been used in the derivation of Eq. (4). In other words the solution of Eq. (4) for the pressure, along with Eq. (2)

and (3) for the velocity field, does not guarantee in any way that the computed velocity field will be divergence-free.

To satisfy the continuity equation (1), the time-dependent term in the right-hand side of Eq. (4) is discretized in time as

$$\frac{\partial D}{\partial t}=\frac{D(t+\Delta t)-D(t)}{\Delta t}, \tag{5}$$

where Δt is the time increment. As suggested by Harlow and Welch [2], the dilation at the $(t+\Delta t)$ time level is set to zero in order to enforce the continuity equation, while the dilation at the (t) time level is retained. Thus, Eq. (5) reduces to

$$\frac{\partial D}{\partial t}=-\frac{D(t)}{\Delta t}. \tag{6}$$

It is important to stress here that the same temporal discretization must be used for the unsteady terms in the momentum equations (2) and (3). By incorporating Eq. (6) into Eq. (4), we obtain

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}=-\left(\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}\right)+\frac{D(t)}{\Delta t}. \tag{7}$$

Equation (7) is the pressure Poisson equation which has been used by researchers to resolve incompressible flows on staggered [2] and non-staggered grids [5–11].

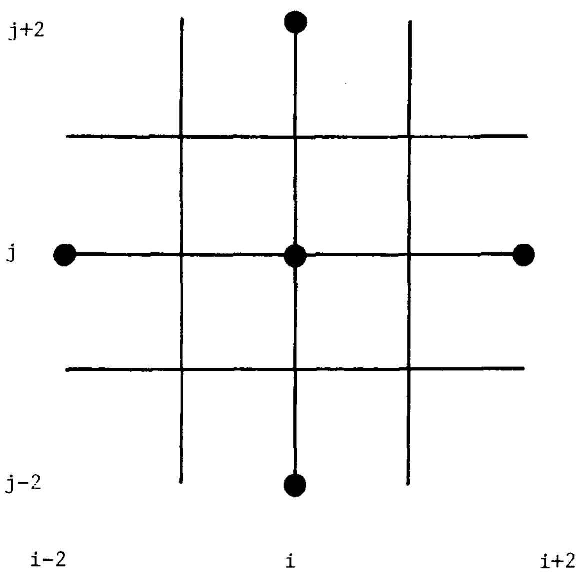

FIG. 1. Finite-difference grid.

To see how the solution of Eq. (7) guarantees the satisfaction of the continuity equation at convergence we should compare Eq. (4) with Eq. (7). This comparison reveals that the pressure Poisson equation (7) can be viewed as an evolution equation for the dilation D, namely Eq. (6). Moreover, the solution of Eq. (6) is an exponential decay in time as

$$D(t) = D_0 e^{-t/\Delta t}, \tag{8}$$

where D_0 is the initial dilation. Thus, the solution of Eq. (7) for the pressure, along with Eqs. (2) and (3) for u and v, guarantees that the initial dilation D_0 will decay to zero as $t \to \infty$ (i.e., as a steady state is approached). We will not pursue further the analysis of the relation between the pressure and the continuity equations in continuum form—a subject which has been investigated by Gresho and Sani [10].

Discretization of the Governing Equations

Using the Euler-explicit temporal discretization scheme for the time derivatives and central second-order accurate finite difference formulas for the spatial derivatives, the system of the governing equations (7), (2), and (3) can be discretized as (see Fig. 1)

$$\begin{aligned}&\frac{P^n_{i+1,j} - 2P^n_{i,j} + P^n_{i-1,j}}{\Delta x^2} + \frac{P^n_{i,j+1} - 2P^n_{i,j} + P^n_{i,j-1}}{\Delta y^2}\\ &\quad = -\left(\frac{\xi^n_{i+1/2,j} - \xi^n_{i-1/2,j}}{\Delta x} + \frac{\eta^n_{i,j+1/2} - \eta^n_{i,j-1/2}}{\Delta y}\right) + \frac{D^n_{i,j}}{\Delta t}\end{aligned} \tag{9}$$

$$u^{n+1}_{i,j} = u^n_{i,j} - \Delta t\left(\frac{P^n_{i+1,j} - P^n_{i-1,j}}{2\Delta x} + \xi^n_{i,j}\right) \tag{10}$$

$$v^{n+1}_{i,j} = v^n_{i,j} - \Delta t\left(\frac{P^n_{i,j+1} - P^n_{i,j-1}}{2\Delta y} + \eta^n_{i,j}\right), \tag{11}$$

where

$$\xi_{i,j} = \left[u_{i,j}\,\delta_x + v_{i,j}\,\delta_y - \frac{1}{\mathrm{Re}}\,(\delta_{xx} + \delta_{yy})\right] u_{i,j} \tag{12a}$$

$$\eta_{i,j} = \left[u_{i,j}\,\delta_x + v_{i,j}\,\delta_y - \frac{1}{\mathrm{Re}}\,(\delta_{xx} + \delta_{yy})\right] v_{i,j}. \tag{12b}$$

The first- and second-order operators (δ_x, δ_y) and (δ_{xx}, δ_{yy}) are the central second-order finite-difference approximations for the ($\partial/\partial x$, $\partial/\partial y$) and ($\partial^2/\partial x^2$, $\partial^2/\partial y^2$) derivatives, respectively. The pressure equation was discretized according to the consistent finite difference method proposed by Abdallah [6].

Equation (9) is solved for $P^n_{i,j}$, given $u^n_{i,j}$ and $v^n_{i,j}$, and then the velocity field is updated using Eq. (10) and (11). At this point of our discussion, it seems important to pose the following question: Does the solution of Eqs. (9), (10), and (11) guarantee that the computed velocity field will be divergence-free in the discrete computational space? In order to answer this question, we should first understand how the discrete continuity equation is modeled in the pressure equation (9). This point can be made very clear if we look at the derivation of Eq. (9) from a different point of view. Let us consider, for example, the following discrete approximation of the continuity equation (1) which we seek to satisfy at the time level $t + \Delta t$ $(n+1)$:

$$\frac{u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}}{\Delta x} + \frac{v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}}{\Delta y} = 0. \tag{13}$$

The pressure Poisson equation (9) can be derived from Eq. (13) if we employ the x- and y-momentum equations at the nodes $(i \pm 1/2, j)$ and $(i, j \pm 1/2)$. Expressions for $u^{n+1}_{i\pm 1/2,j}$ and $v^{n+1}_{i,j\pm 1/2}$ are obtained as follows:

$$u^{n+1}_{i\pm 1/2,j} = u^{n}_{i\pm 1/2,j} - \Delta t \left(\pm \frac{P_{i\pm 1,j} - P_{i,j}}{\Delta x} + \xi_{i\pm 1/2,j} \right)^{n} \tag{14a}$$

$$v^{n+1}_{i,j\pm 1/2} = v^{n}_{i,j\pm 1/2} - \Delta t \left(\pm \frac{P_{i,j\pm 1} - P_{i,j}}{\Delta y} + \eta_{i,j\pm 1/2} \right)^{n}. \tag{14b}$$

Substituting Eqs. (14a) and (14b) into Eq. (13) we obtain

$$\begin{aligned} &\frac{1}{\Delta x}\left[\left(\frac{P_{i+1,j} - P_{i,j}}{\Delta x} + \xi_{i+1/2,j}\right) - \left(\frac{P_{i,j} - P_{i-1,j}}{\Delta x} + \xi_{i-1/2,j}\right)\right]^{n} \\ &\quad + \frac{1}{\Delta y}\left[\left(\frac{P_{i,j+1} - P_{i,j}}{\Delta y} + \eta_{i,j+1/2}\right) - \left(\frac{P_{i,j} - P_{i,j-1}}{\Delta x} + \eta_{i,j-1/2}\right)\right]^{n} \\ &= \frac{1}{\Delta t}\left[\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y}\right]^{n}. \end{aligned} \tag{15}$$

It takes some simple algebra to show that Eq. (15) is identical to Eq. (9). Notice, that the right-hand side of Eq. (15) is the divergence of the velocity field while its left-hand side consists of the x- and y-components of the steady state form of the momentum equation computed at the nodes $(i \pm 1/2, j)$ and $(i, j \pm 1/2)$, respectively. Thus, if the momentum equations are driven to steady state at these nodes, the left-hand side of Eq. (15) will eventually approach zero and so does the divergence of the velocity field. Unfortunately, on a non-staggered grid, the momentum equations are not driven to zero at $(i \pm 1/2, j)$ and $(i, j \pm 1/2)$ nodes but they are at the (i, j) nodes (see Eqs. (10) and (11)). This inconsistency between the discrete momentum and continuity equations prevents the discrete divergence of the velocity (right-hand side of Eq. (15)) from approaching zero. This observation, raises the following question: what is the size of the mass source (error in D) which Eq. (15) introduces into the flow field? We answer this question in the following section.

Estimation of the Error in the Discrete Continuity

To simplify the algebra, we perform the following analysis using the Euler rather than the Navier–Stokes equations. So, we drop the viscous terms from Eqs. (9), (10), and (11). This simplification does not alter significantly the generality of our results.

For the sake of convenience, let us introduce the following notations:

$$f = \frac{\partial P}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} \tag{16a}$$

and

$$h = \frac{\partial P}{\partial y} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}. \tag{16b}$$

Using the above notations, the pressure equation (9) (or (15)) can be written as follows:

$$\frac{f^n_{i+1/2,j} - f^n_{i-1/2,j}}{\Delta x} + \frac{h^n_{i,j+1/2} - h^n_{i,j-1/2}}{\Delta y} = \frac{D^n_{i,j}}{\Delta t}. \tag{17}$$

To express $f_{i\pm 1/2,j}$ and $h_{i,j\pm 1/2}$ in terms of $f_{i,j}$ and $h_{i,j}$ we use Taylor series expansion around the points $(i \pm 1/2, j)$ and $(i, j \pm 1/2)$, respectively.

With reference to Fig. 1, we obtain

$$f_{i\pm 1/2,j} = \tfrac{1}{2}(f_{i\pm 1,j} + f_{i,j}) + ERX^{\pm} \tag{18a}$$

$$h_{i,j\pm 1/2} = \tfrac{1}{2}(h_{i,j\pm 1} + h_{i,j}) + ERY^{\pm}, \tag{18b}$$

where ERX and ERY contain the higher order terms of the series. These terms can be expressed in terms of the primitive variables u, v, and p using Eqs. (16a) and (16b) for f and h. To do so we rewrite Eqs. (18) in the form

$$ERX^{\pm} = f_{i\pm 1/2,j} - \tfrac{1}{2}(f_{i\pm 1,j} + f_{i,j}) \tag{19a}$$

$$ERY^{\pm} = h_{i,j\pm 1/2} - \tfrac{1}{2}(h_{i,j\pm 1} + h_{i,j}). \tag{19b}$$

By substituting the finite-difference approximations of f and h in Eq. (19), one obtains

$$\begin{aligned} ERX^{\pm} = {} & \mp u_{i\pm 1,j}(u_{i\pm 2,j} - 2u_{i\pm 1,j} + u_{i,j})/4\Delta x \\ & \pm u_{i,j}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/4\Delta x \\ & + v_{i,j}(u_{i\pm 1,j+1} - u_{i\pm 1,j-1} - u_{i,j+1} + u_{i,j-1})/8\Delta y \\ & + v_{i\pm 1,j}(-u_{i\pm 1,j+1} + u_{i\pm 1,j-1} + u_{i,j+1} - u_{i,j-1})/8\Delta x \\ & \mp (P_{i\pm 2,j} - 3P_{i\pm 1,j} + 3P_{i,j} - P_{i\pm 1,j})/4\Delta x. \end{aligned} \tag{20}$$

A similar expression for $ERY^{\pm}$ can be obtained using the same method.

As the solution of the system of Eqs. (10), (11), and (17) approaches the steady state solution, the values of $f_{i,j}$ and $h_{i,j}$ approach zero at all the grid points (i, j). Then Eqs. (18) reduce to

$$f_{i \pm 1/2, j} = ERX^{\pm} \tag{21a}$$

$$h_{i, j \pm 1/2} = ERY^{\pm}. \tag{21b}$$

Upon substitution of Eqs. (21) into Eq. (17), we obtain at steady state the value of the dilation

$$D_{i,j} = \Delta t \left(\frac{ERX^{+} - ERX^{-}}{\Delta x} + \frac{ERY^{+} - ERY^{-}}{\Delta y} \right). \tag{22}$$

In order to express $D_{i,j}$ in terms of the dependent variables u, v, and p, we incorporate Eq. (20) and the similarly derived equation for $ERY^{\pm}$ into Eq. (22),

$$\begin{aligned} D_{i,j} = -\frac{\Delta t}{4} [&\Delta x^2 \delta_{xxxx} P + \Delta y^2 \delta_{yyyy} P \\ &+ \Delta x^2 \delta_{xx}(u \delta_{xx} u) + \Delta y^2 \delta_{yy}(v \delta_{yy} v) \\ &+ \text{cross derivative terms}], \end{aligned} \tag{23}$$

where

$$\delta_{xxxx} P = (P_{i+2,j} - 4P_{i+1,j} + 6P_{i,j} - 4P_{i-1,j} + P_{i-2,j})/\Delta x^4 \tag{23a}$$

and

$$\delta_{yyyy} P = (P_{i,j+2} - 4P_{i,j+1} + 6P_{i,j} - 4P_{i,j-1} + P_{i,j-2})/\Delta y^4. \tag{23b}$$

The right-hand side of Eq. (22) is the mass source (error in the discrete continuity) which is introduced into the flow field when Eq. (9) is used to compute the pressure. Clearly, on a non-staggered grid the discrete continuity is satisfied up to a term proportional to the fourth-order spatial derivatives of the pressure and the velocity components. We should mention here that the inclusion of the viscous terms in our analysis would have only introduced higher order terms in Eq. (22) without altering its generality. In any case, the contribution of the viscous terms decreases as the Reynolds number increases.

Equation (22) reveals the following very interesting aspects of the pressure Poisson formulation on a non-staggered grid:

(i) The pressure Poisson method satisfies the discrete continuity equation almost to the same accuracy as the artificial compressibility method does. Recall that a fourth-order artificial dissipation term is explicitly added to the pseudo-compressible continuity equation of the artificial compressibility method [4] to stabilize the numerical solution. In the pressure Poisson formulation, the artificial dissipation is "implicitly" added to the continuity equation because of the way the pressure equation is discretized on a non-staggered grid.

(ii) Our experience with the pressure Poisson method [6, 11] and other researchers' experience with the artificial compressibility method [4] show that the dilation can be quite large in regions of the flow field where high pressure gradients occur. High pressure gradients result in high values of the fourth-order pressure derivatives (artificial dissipation term) and, thus, a significant mass source is generated in the continuity equation. Fortunately, the dilation in the Poisson formulation is explicitly dependent on the square of the grid spacing and the time increment. Therefore, to control the errors in the discrete continuity, we strongly recommend the use of fine grid in regions of high pressure gradients.

In conclusion, careful derivation of the proper discrete pressure equation is very important for accurate incompressible flow solutions (see also Ref. [10]).

Direct Derivation of the Discrete Pressure Equation

The previous discussion suggests that, in order to satisfy the discrete continuity equation, the pressure equation should be derived from the following discrete approximation of Eq. (1):

$$\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2\Delta x}+\frac{v_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}}{2\Delta y}=0. \tag{24}$$

Using Eqs. (10) and (11) into Eq. (24) we obtain the discrete pressure equation

$$\begin{aligned}&\frac{1}{2\Delta x}\left[\left(\frac{P_{i+2,j}-P_{i,j}}{2\Delta x}+\xi_{i+1,j}\right)-\left(\frac{P_{i,j}-P_{i-2,j}}{2\Delta x}+\eta_{i-1,j}\right)\right]^n\\&\quad+\frac{1}{2\Delta y}\left[\left(\frac{P_{i,j+2}-P_{i,j}}{2\Delta y}+\eta_{i,j+1}\right)-\left(\frac{P_{i,j}-P_{i,j-2}}{2\Delta y}+\eta_{i,j-1}\right)\right]^n\\&=\frac{1}{\Delta t}D_{i,j}^n\end{aligned} \tag{25a}$$

or

$$\begin{aligned}&\frac{P_{i+2,j}^n-2P_{i,j}^n+P_{i-2,j}^n}{4\Delta x^2}+\frac{P_{i,j+2}^n-2P_{i,j}^n+P_{i,j-2}^n}{4\Delta y^2}\\&\quad=-\left(\frac{\xi_{i+1,j}^n-\xi_{i-1,j}^n}{2\Delta x}+\frac{\eta_{i,j+1}^n-\eta_{i,j-1}^n}{2\Delta y}\right)+\frac{D_{i,j}^n}{\Delta t}.\end{aligned} \tag{25b}$$

By inspecting Eq. (25a), we can easily conclude that, at convergence, its left-hand side will be driven to zero, since it involves the steady state form of the x- and

y-momentum equations at nodes where they are driven to zero. Therefore, Eq. (25) will satisfy the discrete continuity equation (24) to machine zero.

Unfortunately, the discrete pressure equation (25) produces oscillatory solutions for the pressure because of odd–even decoupling. This can be clearly seen by inspecting the discrete Laplace operator in the left-hand side of Eq. (25) which contains either odd or even grid points in the x- or y-direction. Therefore, in two dimensions Eq. (25) gives decoupled solutions for the pressure on the odd–odd, even–even, odd–even, and even–odd grid points. Each solution is a unique solution for the pressure within an arbitrary constant. Interestingly, the odd–odd solution, for example, is actually a solution for the pressure equation on a staggered grid which is a subset of the original non-staggered grid (see Fig. 2). Similar conclusions can be drawn about the other three solutions.

In conclusion, the use of Eq. (9) to compute the pressure will produce a smooth pressure field, but it will not satisfy the discrete continuity equation exactly, while the use of Eq. (25) will produce exactly the opposite result. In other words, on a non-staggered grid it is not possible to satisfy the discrete continuity to machine zero and, at the same time, obtain a smooth pressure field. In general, one has to sacrifice partially the satisfaction of the discrete continuity, since a smooth and physically meaningful pressure field is desired. This point has been discussed in detail by Strikwerda and Nagel [12].

Therefore, in order to modify Eq. (25) so that it produces smooth pressure field, we must set aside the idea of satisfying the discrete continuity to machine accuracy.

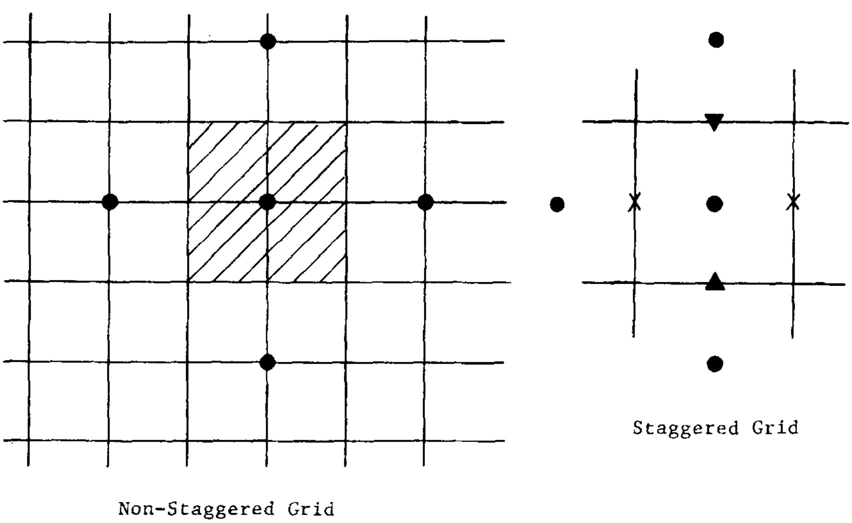

FIG. 2. Equivalence between odd–odd points and staggered grid.

More specifically, we seek to satisfy the discrete continuity equation (13) up to a fraction of the mass source of Eq. (23), as

$$\frac{u^{n+1}_{i+1,j}-u^{n+1}_{i-1,j}}{2\Delta x}+\frac{v^{n+1}_{i,j+1}-v^{n+1}_{i,j-1}}{2\Delta y}$$

$$=-\varepsilon\frac{\Delta t}{4}\left[\Delta x^2\,\delta_{xxxx}(P^n_{i,j})+\Delta y^2\,\delta_{yyyy}(P^n_{i,j})\right], \tag{26}$$

where ε is a positive constant ($0\leqslant\varepsilon\leqslant 1$). Using the momentum equations (10) and (11) in Eq. (26), we obtain the discrete pressure equation,

$$\frac{P_{i+2,j}-2P_{i,j}+P_{i-2,j}}{4\Delta x^2}+\frac{P_{i,j+2}-2P_{i,j}+P_{i,j-2}}{4\Delta y^2}-\frac{\varepsilon}{4}\left[\Delta x^2\,\delta_{xxxx}+\Delta y^2\,\delta_{yyyy}\right]P^n_{i,j}$$

$$=-\sigma^n_{i,j}+\frac{D^n_{i,j}}{\Delta t}, \tag{27}$$

where

$$\sigma_{i,j}=\frac{\xi_{i+1,j}-\xi_{i-1,j}}{2\Delta x}+\frac{\eta_{i,j+1}-\eta_{i,j-1}}{2\Delta y}.$$

It is important to note that for $\varepsilon=0$, Eq. (27) reduces to Eq. (24) while for $\varepsilon=1$ the left-hand side of Eq. (27) reduces to the left-hand side of Eq. (9). Furthermore, both Eqs. (27) and (9) produce a smooth pressure field but they fail to satisfy the discrete continuity to machine zero. However, the error in the discrete continuity which is produced by Eq. (27) is much less than that produced by Eq. (9) because:

(i) There is no error contribution to the mass source (compare Eqs. (23) and (27)) from the source term of the pressure Eq. (27).

(ii) Numerical experiments show that the odd–even decoupling can be removed by using numerical values for $\varepsilon\ll 1$ (see results and discussion section).

Another advantage that Eq. (27) has over Eq. (9) is associated with the computational work (CPU time) required to advance the pressure and velocity fields to the new time level. When Eq. (9) is solved for P along with Eqs. (10) and (11) for u and v, the ξ and η terms need to be discretized twice every time step at $(i\pm 1/2, j\pm 1/2)$ for the pressure equation and at $(i\pm 1, j\pm 1)$ for the momentum equations. This is not the case when Eq. (27) is used, since both the pressure and momentum equations require the calculation of the ξ and η terms at the same nodes. Therefore, we recommend the use of Eq. (27) in the pressure Poisson formulations on non-staggered grids, since it minimizes the error in the discrete continuity ($\varepsilon\ll 1$) and reduces the computational time in the numerical solutions.

Extension of the method to generalized curvilinear coordinates is straightforward with the exception of the source error term in Eq. (27). The error term can be interpreted as the difference between the two finite-difference approximations of the laplace operator given in Eqs. (9) and (25b). Therefore, the error term in non-rectangular variable grids is the difference between the two finite-difference approximations of the transformed laplace operator.

Boundary Conditions for the Pressure Equation

Neumann boundary conditions for the pressure Poisson equation are obtained using the normal component of the momentum equation along the boundary contour. For example, at a boundary $x = \text{const}$:

$$\frac{\partial P}{\partial x} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{28}$$

The boundary condition (28) is applied at one half grid spacing away from the boundary for Eq. (9) [5]. In the case of Eq. (25) the boundary condition (28) is applied at the boundary using one-sided finite difference approximations for the x-derivatives.

The Compatibility Condition

The boundary value problem, consisting of the pressure Poisson equation and the Neumann boundary conditions, has a unique solution if and only if the integral constraint

$$\iint_A \left(-\frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial y} + \frac{D}{\Delta t}\right) dA = \int_S \frac{\partial P}{\partial n}\, ds \tag{29}$$

is satisfied, where n is the outward unit vector normal to the boundary contour S enclosing the solution domain A. Equation (9) satisfies identically the compatibility condition on a non-staggered grid [5]. We will show that Eq. (25) satisfies the integral constraint (29) as well.

For the sake of convenience we use the notation introduced in Eq. (16a) and (16b). Incorporating Eqs. (16) into (27) we obtain

$$\frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{h_{i,j+1} - h_{i,j-1}}{2\Delta y} - \frac{\varepsilon}{4}\left[\Delta x^2\, \delta_{xxxx} + \Delta y^2\, \delta_{yyyy}\right] P_{i,j} = \frac{D_{i,j}}{\Delta t}. \tag{30}$$

The discrete form of the integral constraint (29), when applied to Eq. (30), reads

$$\sum_{i=2}^{im-1} \sum_{j=2}^{jm-1} \left\{\frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{h_{i,j+1} - h_{i,j-1}}{2\Delta y} - \frac{\varepsilon}{4}\left[\Delta x^2\, \delta_{xxxx} + \Delta y^2\, \delta_{yyyy}\right] P_{i,j} - \frac{D_{i,j}}{\Delta t}\right\} = 0, \tag{31}$$

where im and jm are the numbers of the maximum grid points in the x- and y-directions, respectively. Equation (31) can be simplified as

$$\sum_{j=2}^{jm-1} \frac{f_{im,j} + f_{im-1,j} - f_{2,j} - f_{1,j}}{2\Delta x} + \sum_{i=2}^{im-1} \frac{h_{i,jm} + h_{i,jm-1} - h_{i,2} - h_{i,1}}{\Delta t}$$
$$- \sum_{i=2}^{im-1} \sum_{j=2}^{jm-1} \left\{ \frac{\varepsilon}{4} (\Delta x^2 \delta_{xxxx} + \Delta y^2 \delta_{yyyy}) P_{i,j} + \frac{D_{i,j}}{\Delta t} \right\} = 0. \tag{32}$$

Applying the Neumann boundary conditions at $i = 1$ and im ($f_{1,j} = f_{im,j} = 0$) and at $j = 1$ and jm ($h_{i,1} = h_{i,jm} = 0$) and noting that $f_{2,j} = f_{im-1,j} = h_{i,2} = h_{i,jm-1} = 0$ at steady state we can easily see that the first two terms in Eq. (32) vanish identically. Finally, the third term in Eq. (32) is also zero, since the „continuity" equation is locally satisfied at every node. Therefore, Eq. (32) and consequently Eq. (31) are identically satisfied on a non-staggered grid.

Results and Discussion

The driven cavity problem is selected here as the model problem to confirm our analytical developments. Numerical solutions for the momentum equations (10) and (11) are obtained using the Euler explicit scheme and the pressure Poisson equation using the successive over-relaxation method. Both forms of the discrete pressure equations (9) and (27) are used to investigate their ability to satisfy the discrete continuity equation.

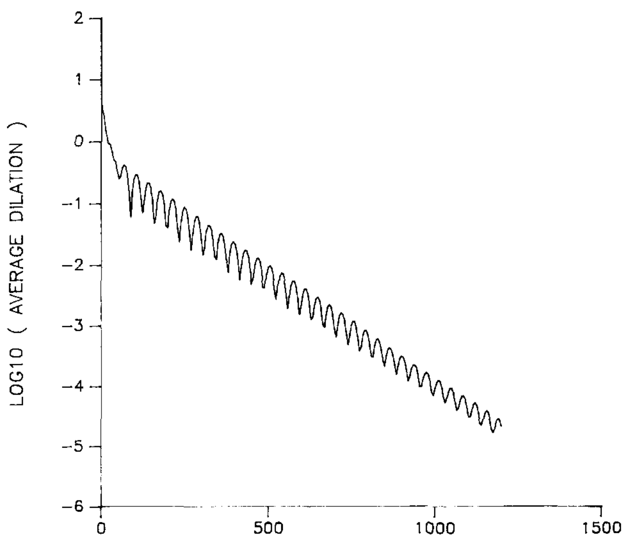

FIG. 3. Convergence of D ($\varepsilon = 0$).

All calculations are performed on uniform non-staggered grids starting from the initial conditions $u_{i,j} = x_{i,j}$, $v_{i,j} = y_{i,j}$, and $P_{i,j} = 0$. The reason for this choice is that we want to explore the capability of the pressure Poisson method to decay an initially high dilation.

The first numerical experiment is conducted on a (31×31) grid using Eq. (27) as the discrete pressure equation. The artificial dissipation parameter ε is set to zero. The time evolution of the logarithm of the average dilation at a node is shown in Fig. 3 Clearly the initially high dilation decays to the machine zero (single precision) at steady state. As we expected, the computed velocity field is smooth, while the pressure field is oscillatory because of odd–even decoupling.

To eliminate the decoupling of the pressure nodes, we conducted a second experiment on the same grid using Eq. (27) with non-zero values of ε. Our results indicate that values of ε as low as 0.1 are sufficient to produce smooth pressure fields. The logarithm of the average dilation at a point is shown in Fig. 4 for $\varepsilon = 0.1$. In the same figure it is also shown the dilation when Eq. (9) is used to compute the pressure on the same grid. As can be seen in Fig. 4, Eq. (27) with $\varepsilon = 0.1$ decays the logarithm of the initial dilation to a steady state value 0.005, while the corresponding value for Eq. (9) is 0.02. The results confirm our analytical developments, since they clearly show that Eq. (9) produces a higher dilation than Eq. (27) with $0 \leqslant \varepsilon \leqslant 1$. We should mention here that the driven cavity problem is a particularly difficult case as far as the satisfaction of the discrete continuity is concerned. The reason for that is the existence of high pressure gradients near the moving wall and

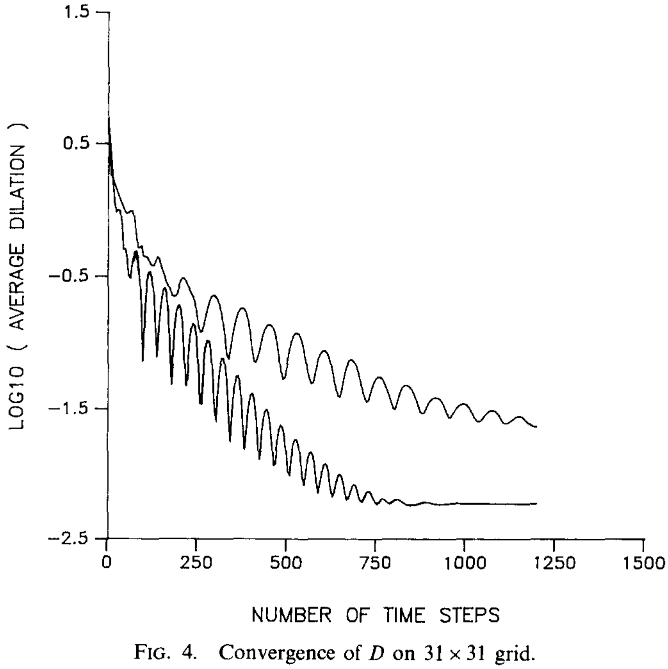

FIG. 4. Convergence of D on 31×31 grid.

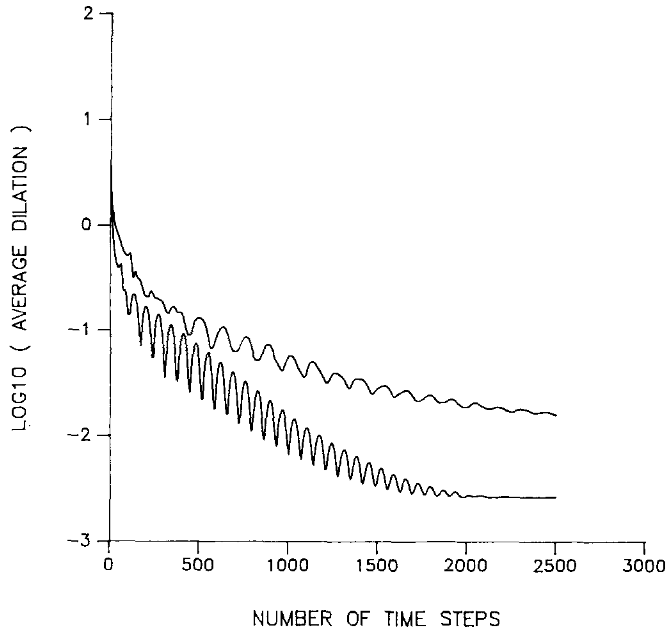

FIG. 5. Convergence of D on 51×51 grid.

the two singular corners. The rapid spatial variation of the pressure in that region results in high values of the dilation (at least an order of magnitude higher than the average value). Recall that the steady state value of the dilation depends upon the fourth-order derivatives of the pressure.

A third numerical experiment is conducted on a (51×51) grid in order to demonstrate the role of the grid refinement on the discrete continuity equation. Figure 5 shows the convergence of the dilation for Eq. (27) with $\varepsilon = 0.1$ and Eq. (9). As expected, Eq. (27) produces a steady state dilation of 0.0024 while the corresponding value for Eq. (9) is 0.014. Comparisons of these values with the corresponding ones on the (31×31) grid confirms the dependence of the dilation on the grid spacing, see Eq. (23).

CONCLUSIONS

We showed that the discrete solutions of incompressible flows, using the classical pressure Poisson formulation on non-staggered grids, do not satisfy the discrete continuity to machine accuracy. We also showed that the error in the discrete continuity is proportional to the fourth-order derivatives of the pressure, the time increment, and the square of the grid spacing. It is interesting to point out that the error is proportional to the explicitly added artificial dissipation term in the pseudo-compressibility method. Since the error is dependent upon the time and space increments, it is recommended that a fine grid, in regions of high pressure gradients, be used in order to minimize the errors in the discrete continuity equation.

We derived a modified discrete pressure Poisson equation which satisfies the discrete continuity equation on non-staggered grids up to a fraction of the dissipation term in the classical pressure Poisson formulation. In addition to minimizing the error in the discrete continuity, the method also reduces the computational work required for the discretization of the pressure equation.

REFERENCES

1. A. J. CHORIN, *J. Comput. Phys.* **2**, 12 (1967).
2. F. H. HARLOW AND J. E. WELCH, *Phys. Fluids* **8**, 2182 (1965).
3. W. Y. SOH AND S. A. BERGER, *Int. J. Numer. Methods* **7**, 733 (1987).
4. S. E. ROGERS, C. O. BOULDER, D. KWAK, AND U. KAUL, "On the Accuracy of the Pseudocompressibility Method in Solving the Incompressible Navier-Stokes Equations," AIAA 85-1689, AIAA 18th Fluid Dynamics and Plasmadynamics and Lasers Conference, Cincinnati, Ohio, July 16–18, 1985 (unpublished).
5. S. ABDALLAH, *J. Comput. Phys.* **70**, 182 (1987).
6. S. ABDALLAH, *J. Comput. Phys.* **70**, 193 (1987).
7. W. R. BRILEY, *J. Comput. Phys.* **14**, 8 (1974).
8. K. N. GHIA, W. L. HANKEY, JR., AND J. K. HODGE, " Study of Incompressible Navier-Stokes Equations in Primitive Variables Using Implicit Nuemrical Technique," AIAA 3rd Computational Fluid Dynamics Conference, Albuquerque, NM, June 27–28, 1977 (unpublished).
9. K. N. GHIA, W. L. HANKEY, JR., AND J. K. HODGE, *AIAA J.* **17**, No. 3, 298 (1979).
10. P. M. GRESHO AND R. L. SANI, *Int. J. Numer. Methods* **7**, 1111 (1987).
11. F. SOTIROPOULOS AND S. ABDALLAH, *J. Comput. Phys.* **87**, 328 (1990).
12. J. C. STRIKWERDA AND Y. M. NAGEL, *J. Comput. Phys.* **78**, 64 (1988).